STRAIGHT LINES AND CIRCLES

Different standard form of the equation of a straight line :

- **General form :** $Ax + By + C = 0$
  where A, B, C are any real numbers not all zero.
- **Gradient (Tangent) form :** $y = mx + c$
  It is the equation of a straight line which cuts off an intercept c on y-axis and makes an angle with the positive direction (anticlockwise) of x-axis such that $\tan\theta = m$. The number m is called slope or the gradient of this line.
- **Intercept form :**
  $$\frac{x}{a} + \frac{y}{b} = 1$$
  It is the equation of straight line which cuts off intercepts a and b on the axis of x and y respectively.
- **Normal form (Perpendicular form) :**
  $x\cos\alpha + y\sin\alpha = p$
  It is the equation of a straight line on which the length of the perpendicular from the origin is p and α is the angle which , this perpendicular makes with the positive direction of x-axis.
- **One point form :**
  $y - y_1 = m(x - x_1)$
  It is the equation of a straight line passing through a given point (x_1, y_1) and having slope m.
- **Parametric equation :**
  $$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$
  It is the equation of a straight line passes through a given point $A(x_1, y_1)$ and makes an angle θ with x-axis.
- **Two points form :**
  $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
  It is the equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2), where $\frac{y_2 - y_1}{x_2 - x_1}$ is its slope.
- **Point of intersection** of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by
  $$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$
- **Angle between two lines :**
  The angle θ between two lines whose slopes are m_1 and m_2 is given by
  $$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$
  If θ is angle between two lines then $\pi - \theta$ is also the angle between them.
- The equation of any straight line parallel to a given line $ax + by + c = 0$ is $ax + by + k = 0$.
- The equation of any straight line perpendicular to a given line, $ax + by + c = 0$ is $bx - ay + k = 0$.
- The equation of any straight line passing through the point of intersection of two given lines $\ell_1 \equiv a_1x + b_1y + c_1 = 0$ and $\ell_2 \equiv a_2x + b_2y + c_2 = 0$ is $\ell_1 + \lambda\ell_2 = 0$
  where λ is any real number, which can be determined by given additional condition in the question.
- The length of perpendicular from a given point (x_1, y_1) to a given line $ax + by + c = 0$ is
  $$\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} = p \text{ (say)}$$
  In particular, the length of perpendicular from origin (0, 0) to the line $ax + by + c = 0$ is $\frac{c}{\sqrt{a^2 + b^2}}$
- **Equation of Bisectors :**
  The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are
  $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$
- **Distance between parallel lines :**
  Choose a convenient point on any of the lines (put x = 0 and find the value of y or put y = 0 and find the value of x). Now the perpendicular distance from this point on the other line will give the required distance between the given parallel lines.

Pair of straight lines :

- The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

- Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

- General equation of second degree in x, y is
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)
This equation represents two straight lines, if
$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\text{or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

and point of intersection of these lines is given by $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$

- The angle between the two straight lines represented by (i) is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between them is given by

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}} \text{ or } 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

Circle:

Different forms of the equations of a circle :

- **Centre radius form :** the equation of a circle whose centre is the point (h, k) and radius 'a' is
$$(x - h)^2 + (y - k)^2 = a^2$$
- **General equation of a circle :** It is given by
$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)
Equation (i) can also be written as
$$|x - (-g)|^2 + |y - (-f)|^2 = |\sqrt{g^2 + f^2 - c}\,|^2$$
which is in centre-radius form, so by comparing, we get the coordinates of **centre** (–g, – f) and **radius** is $\sqrt{g^2 + f^2 - c}$.
- **Parametric Equations of a Circle :**
The parametric equations of a circle
$(x - h)^2 + (y - k)^2 = a^2$ are $x = h + a\cos\theta$ and
$y = k + a\sin\theta$, where θ is a parameter.
- Lengths of intercepts on the coordinate axes made by the circle (i) are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$
- Equation of the circle on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is given by
$$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = 1$$
- If C_1, C_2 are the centres and a_1, a_2 are the radii of two circles, then
(i) The circles touch each other externally, if
$C_1C_2 = a_1 + a_2$
(ii) The circles touch each other internally, if
$C_1C_2 = |a_1 - a_2|$
(iii) The circles intersects at two points, if
$|a_1 - a_2| < C_1C_2 < a_1 + a_2$
(iv) The circles neither intersect nor touch each other, if
$C_1C_2 > a_1 + a_2$ or $C_1C_2 < |a_1 - a_2|$
- Equation of any circle through the point of intersection of two given circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 + \lambda S_2 = 0$ ($\lambda \neq -1$) and λ can be determined by an additional condition.
- Equation of the tangent to the given circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) on it, is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- The straight line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, if $c^2 = a^2(1 + m^2)$ and the point of contact of the tangent $y = mx \pm a\sqrt{1 + m^2}$, is $\left(\frac{\mp ma}{\sqrt{1 + m^2}}, \frac{\pm a}{\sqrt{1 + m^2}}\right)$
- Length of tangent drawn from the point (x_1, y_1) to the circle $S = 0$ is $\sqrt{S_1}$, where
$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
- The equation of pair of tangents drawn from point (x_1, y_1) to the circle
$S = 0$ i.e. $x^2 + y^2 + 2gx + 2fy + c = 0$, is $SS_1 = T^2$, where $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ and S_1 as mentioned above.
- **Chord with a given Middle point :**
the equation of the chord of the circle $S = 0$ whose mid-point is (x_1, y_1) is given by $T = S_1$, where T and S_1 as defined a above.
- If θ be the angle at which two circles of radii r_1 and r_2 intersect, then
$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$
where d is distance between their centres.
Note — Two circles are said to be intersect **orthogonally** if the angle between their tangents at their point of intersection is a right angle i.e.
$r_1^2 + r_2^2 = d^2$ or
$2g_1g_2 + 2f_1f_2 = c_1 + c_2$
- **Radical axis :** The equation of the radical axis of the two circle is $S_1 - S_2 = 0$ i.e.
$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$