TRIGNOMETRIC RATIOS AND IDENTITIES

Some Important Definitions and Formulae :

Measurement of angles : The angles are measured in degrees, grades or in radius which are defined as follows:

Degree : A right angle is divided into 90 equal parts and each part is called a degree. Thus a right angle is equal to 90 degrees. One degree is denoted by 1^o.

A degree is divided into sixty equal parts is called a minute. One minute is denoted by $1'$.

A minute is divided into sixty equal parts and each parts is called a second. One second is denoted by $1''$.

Thus,

1 right angle = 90^o (Read as 90 degrees)

$1^o = 60'$ (Read as 60 minutes)

$1' = 60''$ (Read as 60 seconds).

Grades : A right angle is divided into 100 equal parts and each part is called a grade. Thus a right angle is equal to 100 grades. One grade is denoted by 1^g.

A grade is divided into 100 equal parts and each part is called a minute and is denoted by $1'$.

A minute is divided into 100 equal parts and each part is called a second and is denoted by 1"

Thus,

1 right angled = 100^g (Read as 100 grades)

$1^g = 100'$ (Read as 100 minutes)

$1' = 100''$ (Read as 100 seconds)

Radians : A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

Domain and Range of a Trigono. Function :

If $f : X \to Y$ is a function, defined on the set X, then the domain of the function f, written as Domf is the set of all independent variables x, for which the image f(x) is well defined element of Y, called the co-domain of f.

Range of $f : X \to Y$ is the set of all images f(x) which belongs to Y, i.e.,

Range f = $\{f(x) \in Y : x \in X\} \subseteq Y$

The domain and range of trigonmetrical functions are tabulated as follows :

Trigo. Function	Domain	Range
sin x	R, the set of all the real number	$-1 \le \sin x \le 1$
cos x	R	$-1 \le \cos x \le 1$
tan x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
cosec x	$R - \{n\pi, n \in I\}$	$R - \{x : -1 < x < 1\}$
sec x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - \{x : -1 < x < 1\}$
cot x	$R - \{n\pi, n \in I\}$	R

Relation between Trigonometrically Ratios and identities:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- sin A cosec A = tan A cot A = cos A sec A = 1
- $\sin^2\theta + \cos^2\theta = 1$

  or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$

  or $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta - 1 = \tan^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

  or $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta - 1 = \cot^2\theta$
- Since $\sin^2 A + \cos^2 A = 1$, hence each of sin A and cos A is numerically less than or equal to unity. i.e.

  $|\sin A| \le 1$ and $|\cos A| \le 1$

or $-1 \le \sin A \le 1$ and $-1 \le \cos A \le 1$

Note : The modulus of real number x is defined as $|x| = x$ if $x \ge 0$ and $|x| = -x$ if $x < 0$.

- Since sec A and cosec A are respectively reciprocals of cos A and sin A, therefore the values of sec A and cosec A are always numerically greater than or equal to unity i.e.

  $\sec A \ge 1$ or $\sec A \le -1$

  and $\text{cosec } A \ge 1$ or $\text{cosec } A \le -1$

  In other words, we never have

  $-1 < \text{cosec } A < 1$ and $-1 < \sec A < 1$.

Trigonometrical Ratios for Various Angles :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

Trigonometrical Ratios for Related Angles :

θ	$-\theta$	$\frac{\pi}{2}\pm\theta$	$\pi\pm\theta$	$\frac{3\pi}{2}\pm\theta$	$2\pi\pm\theta$
sin	$-\sin\theta$	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$\cos\theta$
tan	$-\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
cot	$-\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$

Addition and Subtraction Formulae :

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$
- $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$
  $= \cos^2 B - \cos^2 A$
- $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$
  $= \cos^2 B - \sin^2 A$

Formulae for Changing the Sum or Difference into Product :

- $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
- $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

Formulae for Changing the Product into Sum or Difference :

- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

Formulae Involving Double, Triple and Half Angles :

- $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
  $= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
- $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$; $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
  $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$
- $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
  or $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$
- $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
  or $\cos^3\theta = \dfrac{1}{4}(3\cos\theta + \cos 3\theta)$
- $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\left[\theta \neq n\pi + \dfrac{\pi}{6}\right]$

Trigonometrical Ratios for Some Special Angles :

θ	$7\frac{1^\circ}{2}$	15°	$22\frac{1^\circ}{2}$
$\sin\theta$	$\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$
$\cos\theta$	$\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\tan\theta$	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\sqrt{2}-1$

θ	18°	36°
$\sin\theta$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
$\cos\theta$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$
$\tan\theta$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{5-2\sqrt{5}}$

Important Points to Remember :

- Maximum and minimum values of
  $a\sin x + b\cos x$ are $+\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

- $\sin^2 x + \operatorname{cosec}^2 x \geq 2$ for every real x.
- $\cos^2 x + \sec^2 x \geq 2$ for every real x.
- $\tan^2 x + \cot^2 x \geq 2$ for every real x
- If $x = \sec\theta + \tan\theta$, then $\frac{1}{x} = \sec\theta - \tan\theta$
- If $x = \operatorname{cosec}\theta + \cot\theta$, then $\frac{1}{x} = \operatorname{cosec}\theta - \cot\theta$
- $\cos\theta \,.\, \cos 2\theta \,.\, \cos 4\theta \,.\, \cos 8\theta$

  $$\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$$
- $\sin\theta \sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$
- $\cos\theta \cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$
- $\tan\theta \tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$

Conditional Identities :

1. If A + B + C = 180º, then

- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$
- $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$

2. If A + B + C = 180º, then

- $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\sin A + \sin B - \sin C = 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
- $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$

3. If A + B + C = π, then

- $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
- $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$
- $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$

4. If A + B + C = π, then

- $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

5. If x + y + z = π/2, then

- $\sin^2 x + \sin^2 y + \sin^2 z = 1 - 2 \sin x \sin y \sin z$
- $\cos^2 x + \cos^2 y + \cos^2 z = 2 + 2 \sin x \sin y \sin z$
- $\sin 2x + \sin 2y + \sin 2z = 4 \cos x \cos y \cos z$

6. If A + B + C = π, then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
- $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1$
- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

7. (a) For any angles A, B, C we have

- $\sin(A + B + C)$

  $= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
- $\cos(A + B + C)$

  $= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
- $\tan(A + B + C)$

  $$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

(b) If A,B, C are the angles of a triangle, then

$\sin(A + B + C) = \sin\pi = 0$ and

$\cos(A + B + C) = \cos\pi = -1$

then (a) gives

$\sin A \sin B \sin C$

$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C$

and (a) gives

$1 + \cos A \cos B \cos C$

$= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C$

Method of Componendo and Dividendo :

If $\frac{p}{q} = \frac{a}{b}$, then by componendo and dividendo we can write

$$\frac{p-q}{p+q} = \frac{a-b}{a+b} \text{ or } \frac{q-p}{q+p} = \frac{b-a}{b+a}$$

$$\text{or } \frac{p+q}{p-q} = \frac{a+b}{a-b} \text{ or } \frac{q+p}{q-p} = \frac{b+a}{b-a}$$