PERMUTATION AND COMBINATIONS

Permutation :

Definition : The ways of arranging or selecting a smaller or an equal number of persons or objects at a time from a given group of person or objects with due regard being paid to the order of arrangement or selection are called the (different) permutations.

Number of permutations without repetition :

- Arranging n objects, taken r at a time equivalent to filling r places from n things.

r-places :	1	2	3	4	r
Number of choice	n	(n–1)	(n–2)	(n–3)	n–(r – 1)

The number of ways of arranging = The number of ways of filling r places.

$= n(n-1)(n-2) \ldots\ldots (n-r+1)$

$= \dfrac{n(n-1)(n-2)\ldots.(n-r+1)((n-r)!)}{(n-r)!} = \dfrac{n!}{(n-r)!}$

$= {}^nP_r$

- The number of arrangements of n different objects taken all at a time $= {}^nP_n = n!$

(i) ${}^nP_0 = \dfrac{n!}{n!} = 1$; ${}^nP_r = n.\,{}^{n-1}P_{r-1}$

(ii) $0! = 1$; $\dfrac{1}{(-r)!} = 0$ or $(-r)! = \infty \ (r \in N)$

Number of permutations with repetition :

- The number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice, ... upto r times in any arrangement = The number of ways of filling r places where each place can be filled by any one of n objects.

r-places :	1	2	3	4	r
Number of choices :	n	(n)	(n)	(n)	n

The number of permutations = The number of ways of filling r places $= (n)^r$.

The number of arrangements that can be formed using n objects out of which p are identical (and of one kind) q are identical (and of another kind), r are identical (and of another kind) and the rest are distinct is $\dfrac{n!}{p!q!r!}$.

Condition permutations :

- Number of permutations of n dissimilar things taken r at a time when p particular things always occur $= {}^{n-p}C_{r-p}\, r!$.
- Number of permutations of n dissimilar things taken r at a time when p particular things never occur $= {}^{n-p}C_r\, r!$.
- The total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times, is $\dfrac{n(n^r-1)}{n-1}$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$.
- Number of permutation of n different things, taken all at a time, when m specified things never come together is $n! - m! \times (n-m+1)!$.
- Let there be n objects, of which m objects are alike of one kind, and the remaining (n – m) objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\dfrac{n!}{(m!)\times(n-m)!}$.

The above theorem can be extended further i.e., if there are n objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3^{rd} kind; .....; p_r are alike of r^{th} kind such that $p_1 + p_2 + \ldots. + p_r = n$; then the number of permutations of these n objects is $\dfrac{n!}{(p_1!)\times(p_2!)\times\ldots.\times(p_r!)}$.

Circular permutations :

Difference between clockwise and anti-clockwise arrangement : If anti-clockwise and clockwise order of arrangement are not distinct e.g., arrangement of beads in a necklace, arrangement of flowers in garland etc. then the number of circular permutations of n distinct items is $\dfrac{(n-1)!}{2}$.

- Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are taken as different is $\dfrac{{}^np_r}{r}$.

- Number or circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are not different is $\frac{{}^n p_r}{2r}$.

Theorems on circular permutations :

- **Theorem (i) :** The number of circular permutations on n different objects is $(n-1)\,!$.
- **Theorem (ii) :** Then number of ways in which n persons can be seated round a table is $(n-1)\,!$.
- **Theorem (iii) :** The number of ways in which n different beads can be arranged to form a necklace, is $\frac{1}{2}(n-1)!$.

Combinations :

Definition : Each of the different groups or selection which can be formed by taking some or all of a number of objects, irrespective of their arrangements, called a combination.

Notation : The number of all combinations of n things, taken r at a time is denoted by $C(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$. ${}^nC_r$ is always a natural number.

Difference between a permutation and combination :

- In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.
- Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, CBA and CAB correspond to the same combination ABC.

Number of combinations without repetition

The number of combinations (selections or groups) that can be formed from n different objects taken r $(0 \le r \le n)$ at a time is ${}^nC_r = \frac{n!}{r!(n-r)!}$. Also ${}^nC_r = {}^nC_{n-r}$.

Let the total number of selections (or groups) = x. Each group contains r objects, which can be arranged in r ! ways. Hence the number of arrangements of r objects = $x \times (r!)$. But the number of arrangements = ${}^np_r$.

$\Rightarrow x(r\,!) = {}^np_r \Rightarrow x = \frac{n!}{r!(n-r)!} = {}^nC_r$

Number of combinations with repetition and all possible selections :

- The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times.
  = Coefficient of x^r in $(1 + x + x^2 + ... + x^r)^n$
  = Coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$
- The total number of ways in which it is possible to form groups by taking some or all of n things at a time is ${}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$.
- The total number of ways in which it is possible to make groups by taking some or all out of $n = (n_1 + n_2 + ...)$ things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1 + 1)(n_2 + 1) .....\} - 1$.
- The number of selections taking at least one out of $a_1 + a_2 + a_3 + ... + a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on .... a_n are alike (of n^{th} kind) and k are distinct
  $= [(a_1 + 1)(a_2 + 1)(a_3 + 1) .... (a_n + 1)]2^k - 1$

Conditional combinations :

- The number of ways in which r objects can be selected from n different objects if k particular objects are
  (i) Always included = ${}^{n-k}C_{r-k}$
  (ii) Never included = ${}^{n-k}C_r$
- The number of combinations of n objects, of which p are identical, taken r at a time is
  ${}^{n-p}C_r + {}^{n-p}C_{r-1} + ... + {}^{n-p}C_0$, if $r \le p$ and
  ${}^{n-p}C_r + {}^{n-p}C_{r-1} + ... + {}^{n-p}C_{r-p}$, if $r > p$.

Division into groups

- The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}P_n$ or $n\,!\,{}^{n-1}C_{r-1}$ according as blank group are or are not admissible.
- Number of ways in which $m \times n$ different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups)!
  $= \frac{(mn)!n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}$
- If order of group is not important: The number of ways in which mn different things can be divided equally into m groups is $\frac{(mn)!}{(m!)^m m!}$.
- If order of groups is important : The number of ways in which mn different things can be divided equally into m distinct groups is
  $\frac{(mn)!}{(n!)^m m!} \times m\,! = \frac{(mn)!}{(n!)^m}$

Derangement :

Any change in the given order of the things is called a derangement.

If n things form an arrangement in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$n\,!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n.\frac{1}{n!}\right)$