DIFFERENTIAL EQUATIONS

Differential Equation :

An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ... is called differential equation.

Examples :

(1) $\frac{dy}{dx} = 1 + x + y$

(2) $\frac{dy}{dx} + xy = \cot x$

(3) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5 \cos 3x$

(4) $x^2\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = 0$

Order of a Differential Equation :

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of above differential equations are 1, 1, 4 and 2 respectively.

Degree of a Differential Equation :

The degree of the differential equation is the degree of the highest derivative when differential coefficients are free from radical and fraction. For example, the degree of above differential equations are 1, 1, 3 and 2 respectively.

Linear and Non-linear Differential Equation :

A differential equation in which the dependent variable and its differential coefficients occurs only in the first degree and are not multiplied together is called a linear differential equation. The general and n^{th} order differential equation is given below :

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n(x)y + \phi(x) = 0$$

Those equations which are not linear are called non-linear differential equations.

Formation of Differential Equation :

(1) Write down the given equation.

(2) Differentiate it successively with respect to x that number of times equal to the arbitrary constants.

(3) And hence on eliminating arbitrary constants results a differential equation which involves x, y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ .....

Solution of Differential Equation :

A solution of a differential equation is any function which when put into the equation changes it into an identity.

General and particular solution :

The solution which contains a number of arbitrary constant equal to the order of the equation is called general solution by giving particular values to the constants are called particular solutions.

Several Types of Differential Equations and their Solution :

(1) Solution of differential equation

$$\frac{dy}{dx} = f(x) \text{ is } y = \int f(x)dx + c$$

(2) Solution of differential equation

$$\frac{dy}{dx} = f(x)\,g(y) \text{ is } \int \frac{dy}{g(y)} = \int f(x)\,dx + c$$

(3) Solution of diff. equation $\frac{dy}{dx} = f(ax + by + c)$ by

putting $ax + by + c = v$ and $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$

$$\frac{dv}{a + bf(v)} = dx$$

Thus solution is by integrating

$$\int \frac{dv}{a + bf(v)} = \int dx$$

(4) To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, substitute $y = vx$ and so

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Thus $v + x\frac{dv}{dx} = f(v)$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$$

Therefore solution is $\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

Equation reducible to homogeneous form :

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, can be reduced to homogeneous form by adopting the following procedure :

Put $x = X + h,\ y = Y + k,$

so that $\frac{dY}{dX} = \frac{dy}{dx}$

The equation then transformed to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k, the equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

This is a homogeneous equation which can be solved by putting $Y = vX$ and then Y and X should be replaced by $y - k$ and $x - h$.

Special case :

If $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ and $\frac{a}{a'} = \frac{b}{b'} = m$ (say), i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation cannot be solved by the discussed before because the values of h and k given by the equations will be indeterminate.

In order to solve such equations, we proceed as explained in the following example.

Solve $\frac{dy}{dx} = \frac{2x - 6y + 7}{x - 3y + 4} = \frac{2(x - 3y) + 7}{x - 3y + 4}$

$\left\{\text{obviously } \frac{a}{a'} = \frac{b}{b'} = 2\right\}$

Put $x - 3y = v$

$\Rightarrow 1 - 3\frac{dy}{dx} = \frac{dv}{dx}$ (Now proceed yourself)

Solution of the linear differential equation :

$\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x, is

$$ye^{\int P\,dx} = \int\left(Qe^{\int P\,dx}\right)dx + c$$

Where $e^{\int P\,dx}$ is called the integrating factor.

Equations reducible to linear form :

- Bernoulli's equation : A differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone is called Bernoulli's equation.

  Dividing by y^n, we get $y^{-n}\frac{dy}{dx} + y^{-(n-1)}.\ P = Q$

  Putting $y^{-(n-1)} = Y$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dY}{dx}$,

  we get $\frac{dY}{dx} + (1 - n)P.\ Y = (1 - n)Q$

  which is a linear differential equation.

- If the given equations is of the form

  $\frac{dy}{dx} + P.\ f(y) = Q\ .\ g(y)$, where P and Q are functions of x alone, we divide the equation by g(y) and get

  $$\frac{1}{g(y)}\frac{dy}{dx} + P.\frac{f(y)}{g(y)} = Q$$

  Now substitute $\frac{f(y)}{g(y)} = v$ and solve.

Solution of the differential equation :

$\frac{d^2y}{dx^2} = f(x)$ is obtained by integrating it with respect to x twice.