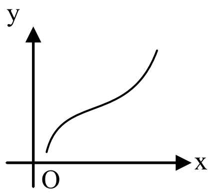
MONOTONICITY

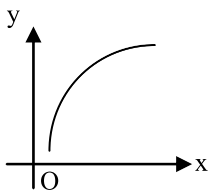
Monotonic Functions :

A function f(x) defined in a domain D is said to be

(i) **Monotonic increasing :**

$$\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$

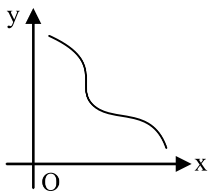


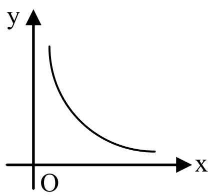


i.e., $\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \not> f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \not< f(x_2) \end{cases} \forall\ x_1, x_2 \in D$

(ii) **Monotonic decreasing :**

$$\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$





i.e., $\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \not< f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \not> f(x_2) \end{cases} \forall\ x_1, x_2 \in D$

A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain.

Note : If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in D$, then f(x) is called strictly increasing in domain D and similarly decreasing in D.

Method of testing monotonicity :

(i) **At a point :** A function f(x) is said to be monotonic increasing (decreasing) at a point x = a of its domain if it is monotonic increasing (decreasing) in the interval (a – h, a + h) where h is a small positive number. Hence we may observer that if f(x) is monotonic increasing at x = a then at this point tangent to its graph will make an acute angle with x-axis where as if the function is monotonic decreasing there then tangent will make an obtuse angle with x-axis. Consequently $f'(a)$ will be positive or negative according as f(x) is monotonic increasing or decreasing at x = a.

So at x = a, function f(x) is

monotonic increasing $\Leftrightarrow f'(a) > 0$

monotonic decreasing $\Leftrightarrow f'(a) < 0$

(ii) **In an interval :** In [a, b], f(x) is

$$\left.\begin{array}{r} \text{monotonic increasing} \Leftrightarrow f'(x) \ge 0 \\ \text{monotonic decreasing} \Leftrightarrow f'(x) \le 0 \\ \text{constant} \Leftrightarrow f'(x) = 0 \end{array}\right\} \forall\ x \in (a, b)$$

Note :

(i) In above results $f'(x)$ should not be zero for all values of x, otherwise f(x) will be a constant function.

(ii) If in [a, b], $f'(x) < 0$ at least for one value of x and $f'(x) > 0$ for at least one value of x, then f(x) will not be monotonic in [a, b].

Examples of monotonic function :

If a functions is monotonic increasing (decreasing) at every point of its domain, then it is said to be monotonic increasing (decreasing) function.

In the following table we have example of some monotonic/not monotonic functions

Monotonic increasing	Monotonic decreasing	Not monotonic
x^3	$1/x,\ x > 0$	x^2
$x\lvert x\rvert$	$1 - 2x$	$\lvert x\rvert$
e^x	e^{-x}	$e^x + e^{-x}$
log x	$\log_2 x$	sin x
sin h x	cosec h x, x > 0	cos h x
[x]	cot hx, x > 0	sec h x

Properties of monotonic functions :

- If f(x) is strictly increasing in some interval, then in that interval, f^{-1} exists and that is also strictly increasing function.
- If f(x) is continuous in [a, b] and differentiable in (a, b), then

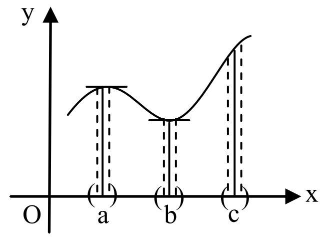
  $f'(c) \ge 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic increasing in [a, b]

  $f'(c) \ge 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic decreasing in [a, b]

- If both $f(x)$ and $g(x)$ are increasing (or decreasing) in $[a, b]$ and gof is defined in $[a, b]$, then gof is increasing.
- If $f(x)$ and $g(x)$ are two monotonic functions in $[a, b]$ such that one is increasing and other is decreasing then gof, it is defined, is decreasing function.

Maximum and Minimum Points :

The value of a function $f(x)$ is said to be maximum at $x = a$ if there exists a small positive number δ such that $f(a) > f(x)$



Also then the point $x = a$ is called a maximum point for the function $f(x)$.

Similarly the value of $f(x)$ is said to be minimum at $x = b$ if there exists a small positive number δ such that

$f(b) < f(x)\ \forall\ x \in (b - \delta, b + \delta)$

Also then the point $x = b$ is called a minimum point for $f(x)$

Hence we find that :

(i) $x = a$ is a maximum point of $f(x)$

$$\Leftrightarrow \begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$$

(ii) $x = b$ is a minimum point of $f(x)$

$$\Leftrightarrow \begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) > 0 \end{cases}$$

(iii) $x = c$ is neither a maximum point nor a minimum point

$$\Leftrightarrow \left\{\begin{matrix} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) \end{matrix}\right\} \text{ have opposite signs.}$$

Where h is a very small positive number.

Note :

- The maximum and minimum points are also known as extreme points.
- A function may have more than one maximum and minimum points.
- A maximum value of a function $f(x)$ in an interval $[a, b]$ is not necessarily its greatest value in that interval. Similarly a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.
- The greatest and least values of a function $f(x)$ in an interval $[a, b]$ may be determined as follows :

  Greatest value = max. $\{f(a), f(b), f(c)\}$

  Least value = min. $\{f(a), f(b), f(c)\}$

  where $x = c$ is a point such that $f'(c) = 0$.
- If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.
- Monotonic functions do not have extreme points.

Conditions for maxima and minima of a function

Necessary condition : A point $x = a$ is an extreme point of a function $f(x)$ if $f'(a) = 0$, provided $f'(a)$ exists. Thus if $f'(a)$ exists, then

$x = a$ is an extreme point $\Rightarrow f'(a) = 0$ or

$f'(a) \neq 0 \Rightarrow x = a$ is not an extreme point

But its converse is not true i.e.

$f'(a) = 0 \not\Rightarrow x = a$ is an extreme point.

For example if $f(x) = x^3$, then $f'(0) = 0$ but $x = 0$ is not an extreme point.

Sufficient condition : For a given function $f(x)$, a point $x = a$ is

- a maximum point if $f'(a) = 0$ and $f''(a) < 0$
- a minimum point if $f'(a) = 0$ and $f''(a) > 0$
- not an extreme point if $f'(a) = 0 = f''(a)$ and $f'''(a) \neq 0$.

Note : If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) = 0$ then the sign of $f^{(4)}(a)$ will determine the maximum or minimum point as above.

Working Method :

- Find $f'(x)$ and $f''(x)$.
- Solve $f'(x) = 0$. Let its roots be a, b, c, ...
- Determine the sign of $f''(x)$ at $x = a, b, c$, ... and decide the nature of the point as mentioned above.

Properties of maxima and minima :

If $f(x)$ is continuous function, then

- Between two equal values of $f(x)$, there lie atleast one maxima or minima.
- Maxima and minima occur alternately. For example if $x = -1, 2, 5$ are extreme points of a continuous function and if $x = -1$ is a maximum point then $x = 2$ will be a minimum point and $x = 5$ will be a maximum point.
- When x passes a maximum point, the sign of dy/dx changes from + ve to – ve, where as when x passes through a minimum point, the sign of $f'(x)$ changes from –ve to + ve.
- If there is no change in the sign of dy/dx on two sides of a point, then such a point is not an extreme point.
- If $f(x)$ is maximum (minimum) at a point $x = a$, then $1/f(x)$, $[f(x) \neq 0]$ will be minimum (maximum) at that point.
- If $f(x)$ is maximum (minimum) at a point $x = a$, then for any $\lambda \in R$, $\lambda + f(x)$, $\log f(x)$ and for any $k > 0$, $k f(x)$, $[f(x)]^k$ are also maxmimum (minimum) at that point.